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Three-Point Phase, Symplectic Measure, and Berry Phase

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It is shown that the geometric phase (Berry phase) around a cycle in the complex projective space of pure states of a quantum mechanical system can be expressed in terms of an elementary three-point phase function which is the simplest manifestation of the complexity of the underlying Hilbert space. In terms of this three-point phase it is possible to construct a geometrically relevant phase function defined mod 4π on the cycles and closely related to the natural symplectic structure of the state space.

Page (1987) has shown that the geometric phase (Berry, 1984; Aharonov and Anandan, 1987, 1990) (Berry phase) associated with an oriented cycle C in the complex projective space $\mathcal{P}$ of pure states of a quantum mechanical system is equal, mod 2π, to the symplectic measure $\int_S \Omega$ of any two-dimensional submanifold S of $\mathcal{P}$ with boundary C, provided that the natural symplectic form Ω of $\mathcal{P}$ (Arnold, 1978; Dubrovin *et al.*, 1990) is suitably normalized. We shall show that $\int_S \Omega$ is directly determined by the simplest significant manifestation of the complexity of the underlying Hilbert space $\mathcal{H}$, namely the elementary angular three-state function ϑ that we shall now consider and call the *three-point phase.*

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be unit representatives in $\mathcal{H}$ of two nonorthogonal pure states α and β: in the complex scalar product $\langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle \equiv \rho \exp(i\phi)$ the angle ϕ has no physical relevance, since it can take any value according to the choice of the representatives. But if three pairwise nonorthogonal states α, β, and γ are considered and one sets $\langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle\langle\boldsymbol{\beta}, \boldsymbol{\gamma}\rangle\langle\boldsymbol{\gamma}, \boldsymbol{\alpha}\rangle \equiv \tau \exp(-i\vartheta)$, the angle ϑ (determined mod 2π) is unaffected by changes of the representatives and depends on the triple (α, β, γ) only (with a change of sign under odd permutations of the states). For this reason the *three-point phase* $\vartheta(\alpha, \beta, \gamma)$ is likely to be physically significant.

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We shall show that around the minimal geodesic triangle $\alpha\beta\gamma$ [i.e., the triangle in $\mathscr{P}$ whose sides are the shortest geodesics—with respect to the Fubini–Study metric (Kobayashi and Nomizu, 1969)—joining the vertices α, β, and γ], the three-point phase is the same thing as the Berry phase. Then, by means of such elementary triangular meshes, we shall express the Berry phase around any element C of a quite general class of cycles. More precisely, from the three-point phase ϑ we shall construct a phase-function $\Phi(C)$, defined mod 4π on the cycles, which turns out to be equal (mod 4π) to the symplectic measure of any two-dimensional submanifold bounded by C. Therefore Φ is also equal, mod 2π, to the Berry phase. Finally, we make some remarks on the geometrical significance of the fact that the phase function Φ is defined mod 4π (rather than mod 2π).

First we compute, by applying Page's result, the Berry phase around the geodesic triangle $\alpha\beta\gamma$.

If two pure states λ and μ are not orthogonal to each other and are represented by the unit vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, the shortest oriented geodesic $\lambda\mu$ from λ to μ in $\mathscr{P}$ (with respect to the Fubini–Study metric) is the map

$$t \to \frac{\boldsymbol{\lambda}+[(\langle\boldsymbol{\mu},\boldsymbol{\lambda}\rangle/|\langle\boldsymbol{\mu},\boldsymbol{\lambda}\rangle|)\boldsymbol{\mu}-\boldsymbol{\lambda}]t}{[1+2t(t-1)(1-|\langle\boldsymbol{\lambda},\boldsymbol{\mu}\rangle|)]^{1/2}} \tag{1}$$

from the unit interval to $\mathscr{P}$ (where the points of $\mathscr{P}$ are identified by their representatives in $\mathscr{H}$).

We introduce a coordinate system in $\mathscr{P}$, adapted to our problem, as follows. First, we note that once a definite unit representative $\boldsymbol{\alpha}$ of α has been arbitrarily chosen, a normalized representative $\boldsymbol{\lambda}$ of any other point λ of $\mathscr{P}$ can be uniquely selected by requiring that $\langle\boldsymbol{\lambda},\boldsymbol{\alpha}\rangle$ be positive, provided that λ and α are not orthogonal. With this choice of the representatives we then consider an orthonormal basis $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2 \ldots)$ of $\mathscr{H}$ with first element $\mathbf{e}_0 \equiv \boldsymbol{\alpha}$, second element $\mathbf{e}_1$ such that $\boldsymbol{\beta}$ is a real linear combination of $\mathbf{e}_0$ and $\mathbf{e}_1$, and third vector $\mathbf{e}_2$ such that $\boldsymbol{\gamma}$ is a (generally complex) linear combination of $\mathbf{e}_0$, $\mathbf{e}_1$, and $\mathbf{e}_2$ and such that $\langle\boldsymbol{\gamma},\mathbf{e}_2\rangle$ is real. Then every point λ of $\mathscr{P}$ not orthogonal to α is uniquely identified by the independent complex components $z_1, z_2, z_3 \ldots$ of its representative $\boldsymbol{\lambda}$ with respect to this basis:

$$\boldsymbol{\lambda} = \left(1-\sum_h \bar{z}_h z_h\right)^{1/2} \mathbf{e}_0 + \sum_h z_h \mathbf{e}_h$$

Setting $z_h \equiv x_h + iy_h$, we get a system of real coordinates $(x_1, y_1, x_2, y_2, \ldots)$ that we shall also use.

These coordinates turn out to be canonical for the natural symplectic form Ω of $\mathscr{P}$, up to a factor which depends on the normalization of Ω. Following Page, we shall adopt the normalization which makes the period of Ω equal to 4π, namely $\Omega = 2\sum_h dy_h \wedge dx_h \equiv i\sum_h d\bar{z}_h \wedge dz_h$. Since the points on the geodesic arcs $\alpha\beta$, $\beta\gamma$, and $\gamma\alpha$ correspond to vectors of $\mathscr{H}$ all lying in the subspace generated by $\mathbf{e}_0$, $\mathbf{e}_1$, and $\mathbf{e}_2$, the arcs belong to the real finite-dimensional submanifold of $\mathscr{P}$ on which $x_h = 0$ and $y_h = 0$ for $h > 2$, and the integral of Ω on any two-dimensional surface bounded by the triangle $\alpha\beta\gamma$ and lying in the domain of the coordinates can be evaluated, via Stokes' theorem, by integrating the 1-form

$$\Gamma \equiv \sum_{h=1}^{2} (y_h\, dx_h - x_h\, dy_h) \equiv \frac{i}{2} \sum_{h=1}^{2} (\bar{z}_h\, dz_h - z_h\, d\bar{z}_h)$$

along the contour

$$\iint \Omega = \oint \Gamma \tag{2}$$

We must compute the line integral on the right-hand side (which is coordinate-independent, though the 1-form Γ and, with it, the separate contribution of each side of the contour are not).

According to our conventions, $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$ and $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle$ are positive, while $\langle \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$ is in general complex, so that we can set $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = c$, $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle = b$, and $\langle \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle = a \exp(-i\vartheta)$, where a, b, and c are positive and ϑ is the three-point phase defined above. According to (1), the side $\alpha\beta$ of our geodesic triangle has the coordinate representation

$$x_1(t) = \frac{(1-c^2)^{1/2} t}{[1 + 2t(t-1)(1-c)]^{1/2}}, \qquad y_1(t) = 0, \qquad x_2(t) = 0, \qquad y_2(t) = 0$$

and does not contribute to the line integral. The side $\gamma\alpha$ has the coordinate representation

$$x_1(t) = \frac{a \cos \vartheta - bc}{(1-c^2)^{1/2}}\, T(b, t), \qquad y_1(t) = \frac{a \sin \vartheta}{(1-c^2)^{1/2}}\, T(b, t)$$

$$x_2(t) = \frac{(1 - a^2 - b^2 - c^2 + 2abc \cos \vartheta)^{1/2}}{(1-c^2)^{1/2}}\, T(b, t), \qquad y_2(t) = 0$$

where we have set $T(b, t) \equiv (1-t)/[1 + 2t(t-1)(1-b)]^{1/2}$, from which one gets $\int_{\gamma\alpha} \sum_{h=1}^{2} (x_h\, dy_h - y_h\, dx_h) = 0$.

In order to compute the contribution of the side $\beta\gamma$, it is convenient to introduce a second coordinate system $(x'_1, y'_1, x'_2, y'_2, \ldots)$ constructed exactly like $(x_1, y_1, x_2, y_2, \ldots)$ except for the replacement of α with β, β with γ, and γ with α. Denoting by $(\mathbf{e}'_0, \mathbf{e}'_1, \mathbf{e}'_2 \ldots)$ the orthonormal basis related to the new coordinates as $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2 \ldots)$ was related to the old ones, the generic state λ is now parametrized by the real and imaginary parts of the independent components $(z'_1, z'_2, \ldots)$ of the representative

$$\boldsymbol{\lambda}' = \exp(i\psi)\, \boldsymbol{\lambda} \tag{3}$$

where ψ must be chosen such that $\langle \boldsymbol{\lambda}', \boldsymbol{\beta} \rangle$ be positive, i.e.,

$$\psi(\lambda) = \operatorname{Arg}\langle \boldsymbol{\lambda}, \boldsymbol{\beta} \rangle \tag{4}$$

By decomposing the left-hand side of (3) with respect to the new basis and the right-hand side with respect to the old one, we get

$$l'\boldsymbol{\beta} + z_1'\mathbf{e}_1' + z_2'\mathbf{e}_2' = \exp(i\psi)\,(l\boldsymbol{\alpha} + z_1\mathbf{e}_1 + z_2\mathbf{e}_2)$$

where $l \equiv (1 - \bar{z}_1 z_1 - \bar{z}_2 z_2)^{1/2}$ and l' is defined analogously. The scalar products of this relation with $\mathbf{e}_1$ and $\mathbf{e}_2$ yield $z_1 = \exp(-i\psi)\, L_1$ and $z_2 = \exp(-i\psi)\, L_2$, where $L_h \equiv l'\langle \mathbf{e}_h, \beta\rangle + \langle \mathbf{e}_h, \mathbf{e}_1'\rangle z_1' + \langle \mathbf{e}_h, \mathbf{e}_2'\rangle z_2'$ and, according to (4), ψ can be regarded as a well-determined function of z_1' and z_2'. From these relations one gets

$$\sum_1^2 (\bar{z}_h\, dz_h - z_h\, d\bar{z}_h) = \sum_1^2 (\bar{L}_h\, dL_h - L_h\, d\bar{L}_h - 2i\bar{L}_h L_h\, d\Psi) \tag{5}$$

Now, in the primed coordinates the geodesic $\beta\gamma$ has the representation

$$x_1'(t) = \frac{(1-a^2)^{1/2} t}{[1 + 2t(t-1)(1-a)]^{1/2}}, \qquad y_1'(t) = 0, \qquad x_2'(t) = 0, \qquad y_2'(t) = 0$$

and the restriction $\Gamma_{\beta\gamma}$ of the 1-form Γ to this side of the triangle reduces, after some calculation, to the following expression:

$$\Gamma_{\beta\gamma} = \frac{bc \sin\vartheta}{(1-a)^{1/2}} [2(1 - x_1'^2)^{1/2}\, dx_1' - d(x_1'(1 - x_1'^2)^{1/2})]$$
$$- \frac{bc \sin\vartheta\, dt}{1 + 2t(t-1)(1-a)} + d\psi$$

After integration from β to γ all terms except the last one cancel out, so that $\int_{\beta\gamma} \Gamma = \psi(\gamma) - \psi(\beta)$, which is just ϑ on account of (4). Summing up, we obtain $\oint \Gamma = \vartheta$, which shows that the Berry phase around a geodesic triangle indeed coincides with the three-point phase ϑ.

We now use this result to construct, from ϑ, a phase function $\Phi(c)$ defined, mod 4π, on a restricted class $\mathscr{C}_0$ of one-dimensional cycles in $\mathscr{P}$ that we shall call "small cycles." Next we shall extend Φ to a wider class $\mathscr{C}$.

By *small cycle* we mean a closed oriented curve $t \to c(t)$ $[0 \leq t \leq 1$, $c(0) = c(1)]$ entirely contained in some open ball of $\mathscr{P}$ of radius $\pi/2$ with respect to the distance function determined by the Fubini–Study metric, the metric being normalized so that the length of the closed geodesics is 2π. Thus the ball contains no pair of conjugate points, i.e., no pair of points representing orthogonal states, and it lies in the domain of some local coordinate system.

For any finite partition of the unit interval determined by the numbers $t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$, let us denote by $\beta_i \equiv c(t_i)$ the corresponding points on the small cycle c. Let us choose any point α in a ball containing c, and set $\sigma \equiv \sum_{i=0}^{n-1} \bar{\vartheta}(\alpha, \beta_1, \beta_{i+1})$, where, for each geodesic triangle $\alpha\beta_i\beta_{i+1}$, $\bar{\vartheta}(\alpha, \beta_i, \beta_{i+1})$ denotes the unique representative of the three-point phase ϑ such that $-\pi \le \bar{\vartheta} < \pi$. Moreover, let us denote by s any oriented 2-manifold bounded by c and lying in the ball, and by s_i any oriented 2-manifold bounded by the triangle $\beta_i\alpha\beta_{i-1}$, also in the ball. From our previous result we have $\sigma = -\int_u \Omega$, where $u \equiv \bigcup_{i=0}^{n-1} s_i$ is the union of the "triangular" surfaces s_i. On the other hand, it is possible to choose an additional oriented 2-manifold ε in the ball, bounded by c and by the geodesic polygon $\beta_1\beta_2 \ldots \beta_n$, in such a way that its union with s and u forms a two-dimensional cycle (Figure 1). From the fact that the integral of Ω over such cycles vanishes mod 4π (4π being the period of Ω with our normalization), it is not difficult to prove that σ is independent of the choice of the ball containing c and of the point α in the ball, and that

$$\lim_{\Delta \to 0} \sigma = \int_S \Omega \tag{6}$$

where the limit is taken over the partitions of the unit interval and, for each partition, $\Delta \equiv \max|t_{i+1} - t_i|$. Thus, we can set

$$\Phi(c) \equiv \lim_{\Delta \to 0} \sigma \tag{7}$$

and the function Φ so defined is well determined mod 4π on the class $\mathscr{C}_0$ of small cycles.

The function Φ defined by (7) on the set of small cycles is directly determined by the three-point phase. On account of its property (6), it is

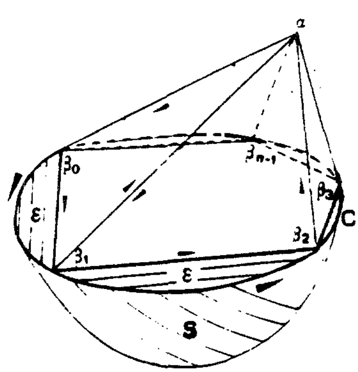

Fig. 1. The approximation of the phase function around a small cycle by a sum of three-point phases.

easy to extend Φ from small cycles to the boundary C of any oriented 2-manifold S, provided that S is decomposable, as sketched in Figure 2, into portions s_h bounded by small cycles c_h. One can simply set $\Phi(C) = \sum_h \Phi(c_h)$, and it is an immediate consequence of (6) that, with this definition, $\Phi(C)$ is well determined mod 4π, irrespective of the particular choice of the surface S bounded by C and of the particular decomposition of S.

The fact that the phase function Φ is defined mod 4π (rather than mod 2π) is geometrically significant, in the sense that for any number ϕ such that $0 \leq \phi < 2\pi$ the set of cycles C such that $\Phi(C)$ is represented by ϕ and the set of cycles C' such that $\Phi(C')$ is represented by $\phi + 2\pi$ are distinct.

To visualize this point, let us consider the complex projective space $\mathcal{P}$ of lowest dimension, namely CP^1. In this case, with our normalization of the metric, $\mathcal{P}$ is just the 2-sphere of curvature 1, and with our normalization of Ω the symplectic measure of the whole space is 4π. For any ϕ in the range $0 \leq \phi < 2\pi$, let C be the generic cycle of a class such that $\Phi(C)$ is represented by ϕ: then one of the connected regions of the sphere bounded by C has area ϕ, and the complementary region has area $4\pi - \phi$. Similarly, if ϕ is replaced by $\phi' \equiv \phi + 2\pi$, the cycles C' of the class such that $\Phi(C')$ is represented by ϕ' divide the sphere into two complementary regions with areas $\phi + 2\pi$ and $2\pi - \phi$. Obviously the two classes of cycles are disjoint.

More particularly, for ϕ in the range $0 \leq \phi < 4\pi$, let α, β, and γ be pairwise nonconjugate points such that the oriented area of one of the regions enclosed by the minimal geodesic triangle $\alpha\beta\gamma$ is ϕ (Figure 3a). If $\alpha\beta\gamma$ is replaced by a (nonminimal) geodesic triangle with the same vertices but one of the sides replaced by the maximal geodesic between its endpoints (Figure 3b), the corresponding area ϕ' enclosed by the new triangle is $\phi + 2\pi$ if $\phi < 2\pi$ and $\phi - 2\pi$ if $\phi \geq 2\pi$. (In the special case of three points α, β, and γ on the same great circle, one of the two triangles determines a cycle enclosing a vanishing area, while the other is the whole oriented great

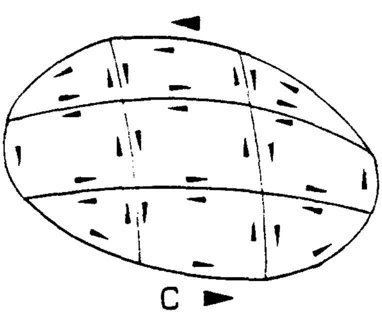

Fig. 2. The relation between a generic cycle and small cycles.

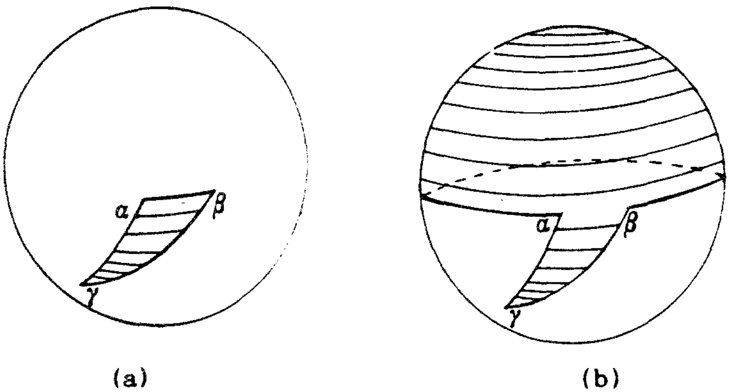

Fig. 3. (a) A minimal geodesic triangle with phase ϕ. (b) A geodesic triangle with the same vertices, but with phase $\phi+2\pi$.

circle, which encloses an area 2π.) In all cases two such distinct geodesic triangles with the same vertices are cycles of quite different kinds, which are distinguished by the phase function Φ.

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